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Algebraic Constraints for Higgs Multiplets

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Abstract

The phenomenon of spontaneous breaking of a gauge symmetry group, provides a number of algebraic constraints which Scalar Higgs mesons have to satisfy. We discuss these constraints and give details for the cases of $SU(2)$, $SU(2) \times U(1)$ and $SU(3)$.

1. *Introduction*

The spontaneous mode of symmetry breaking has been discussed in various connections in high energy physics during the past few years. Recently it has been used as an ingredient in Weinberg's type of gauge theories of electromagnetic and weak interactions. In these gauge theories one supposes that some symmetry group G is spontaneously broken by a scalar Higgs meson $\varphi(x)$. This Higgs meson has to be assigned to some irreducible representation of a chosen gauge symmetry group G. The choice of the representation for $\varphi(x)$ is usually dictated by physical considerations.

However, as pointed out in a recent paper by Lieberman (1973), certain mathematical constraints have also to be satisfied by the chosen representation for φ. Using graphical methods, Lieberman finds that the constraints imply that certain low-dimensional representations of G are not acceptable for classifying $\varphi(x)$, even when, on physical grounds, such representations may appear reasonable.

These mathematical constraints can be formulated in a purely algebraic manner, leading to a straightforward solution of the problem of allowed and forbidden representations for the Higgs mesons. In this paper we discuss these algebraic constraints, illustrating the arguments by the specific cases of $SU(2)$, $SU(2) \times U(1)$ and $SU(3)$ as gauge groups.

2. *Representation of the Higgs Field*

Let G be an n parameter gauge group which is spontaneously broken but such that the vacuum remains invariant under some m parameter subgroup H of G. In that case we write

$$T_a\langle 0|\varphi|0\rangle = 0 \qquad \text{for } a = 1, 2, 3, \ldots, m \tag{2.1}$$

$$T_b\langle 0|\varphi|0\rangle \neq 0 \qquad \text{for } b = 1, 2, \ldots, r = (n-m) \tag{2.2}$$

where the T's are the generators of G. These relations then imply that the representations of G to which the scalar Higgs meson φ may be assigned have to be such that the above constraints are satisfied. Thus if φ is assigned to some p-dimensional representation we must require that

$$T_a\langle\varphi_i\rangle = \langle T_a\varphi_i\rangle = 0 \tag{2.3}$$

$$T_b\langle\varphi_i\rangle = \langle T_b\varphi_i\rangle \neq 0 \tag{2.4}$$

where $i = 1, 2, \ldots, p$; $a = 1, 2, \ldots, m$; $b = 1, 2, \ldots, r$. In the more usual case, H is chosen to be a one-parameter group generated by the electric charge operator Q. This allows the photon to remain massless after the gauge symmetry is spontaneously broken. Here, however, we shall not be bound completely by such physical considerations but will take H to be some arbitrary m-parameter non-abelian subgroup.

Using now the constraints (2.3) and (2.4) we can consider which representations for φ will be allowed or forbidden. The problem is solved by calculating the matrix elements of the generators T_a and T_b and substituting into equations (2.3) and (2.4).

Taking the simple case $G = SU(2)$ and H as the one-parameter subgroup generated by the diagonal operator I_z, the calculation proceeds as follows: We assume the Higgs meson φ assigned to an arbitrary p-dimensional representation of $SU(2)$ so that we can write $\varphi_i = \varphi^{\mathrm{II}_z}$ where I_z is the third component of isospin. The constraint equations then take the form

$$\langle T_z\varphi^{\mathrm{II}_z}\rangle = 0 \tag{2.5}$$

$$\langle T_+\varphi^{\mathrm{II}_z}\rangle \neq 0 \tag{2.6}$$

$$\langle T_-\varphi^{\mathrm{II}_z}\rangle \neq 0 \tag{2.7}$$

Now the matrix elements of these generators are given by

$$T_z\varphi^{\mathrm{II}_z} = I_z\varphi^{\mathrm{II}_z} \tag{2.8}$$

$$T_+\varphi^{\mathrm{II}_z} = \sqrt{[(I - I_z)(I + I_z + 1)]}\,\varphi^{\mathrm{I,I}_z+1} \tag{2.9}$$

$$T_-\varphi^{II_z} = \sqrt{[(I + I_z)(I - I_z + 1)]}\,\varphi^{I,I_z-1} \tag{2.10}$$

Substituting (2.8), (2.9) and (2.10) into equations (2.5), (2.6) and (2.7) we require that

$$I_z < \varphi^{II_z}\rangle = 0 \tag{2.11}$$

$$\sqrt{[(I - I_z)(I + I_z + 1)]}\,\langle \varphi^{I,I_z+1} \rangle \neq 0 \tag{2.12}$$

$$\sqrt{[(I + I_z)(I - I_z + 1)]}\,\langle \varphi^{I,I_z-1} \rangle \neq 0 \tag{2.13}$$

Equation (2.11) implies that the only states with non-zero vacuum expectation value are those with $I_z = 0$. Then equations (2.12) and (2.13) lead to the result that for $I_z \pm 1 = 0$,

$$\sqrt{[(I \mp I_z)(I \pm I_z + 1)]} \neq 0$$

i.e.

$$(I + 1)I \neq 0$$

This is satisfied provided that $I \neq 0$. One then concludes that the algebraic constraint arising from the spontaneous mode of the breaking of the $SU(2)$ symmetry, precludes the Higgs meson from being assigned to the one-dimensional representation of $SU(2)$. All other representations are mathematically acceptable.

Consider next the case where $G = SU(2) \times U(1)$ with the generators $I_z, I_\pm$ and Y. Let H be chosen again as the one-parameter subgroup generated now by the electric charge operator Q. Then the constraint equations take the form

$$(I_z + \tfrac{1}{2}Y) = 0$$

with

$$\langle \varphi^{II_zY} \rangle \neq 0$$

Then for $I_z + 1 = -\frac{1}{2}Y$, we require

$$\sqrt{[(I - I_z)(I + I_z + 1)]} \neq 0$$

This implies that $I \neq \frac{1}{2}Y$ and $I \neq -(Y + 2)$. This means, for example, that if the scalar Higgs meson φ is chosen to be an isodoublet as in the Weinberg model, then its hypercharge Y cannot be chosen as $Y = 1$ or -3.

Next we discuss the case of $SU(3)$ as a gauge group, choosing H to be the isospin $SU(2)$ group. Denoting the generators of $SU(3)$ by $I_z, Y, I_\pm, U_\pm, V_\pm$ and an arbitrary basis operator by φ^{II_zY} and using the following standard relations (Biedenharn, 1962; de Swart, 1966; Mukunda & Pandit, 1965)

$$I_\pm \varphi^{I,I_z,Y} = \sqrt{[(I \mp I_z)(I \pm I_z + 1)]}\,\varphi^{I,I\pm 1,Y}$$

$$V_- \varphi^{II_zY} = b_+ \varphi(I + \tfrac{1}{2}, I_z + \tfrac{1}{2}, Y + 1) + b_- \varphi(I - \tfrac{1}{2}, I_z + \tfrac{1}{2}, Y + 1)$$

$$U_+ \varphi^{II_zY} = c_+ \varphi(I + \tfrac{1}{2}, I_z - \tfrac{1}{2}, Y + 1) + c_- \varphi(I - \tfrac{1}{2}; I_z - \tfrac{1}{2}; Y + 1)$$

where

$$b_+ = \left[\frac{(I+I_z+1)\left(\frac{\lambda-\mu}{3}+I+\frac{1}{2}Y+1\right)\left(\frac{\lambda+2\mu}{3}+I+\frac{1}{2}Y+2\right)\left(\frac{2\lambda+\mu}{3}-I-\frac{1}{2}Y\right)}{2(I+1)(2I+1)}\right]^{1/2}$$

$$b_- = \left[\frac{(I-I_z)\left(\frac{\mu-\lambda}{3}+I-\frac{1}{2}Y\right)\left(\frac{\lambda+2\mu}{3}-I+\frac{1}{2}Y+1\right)\left(\frac{2\lambda+\mu}{3}+I-\frac{1}{2}Y+1\right)}{2I(2I+1)}\right]^{1/2}$$

$$C_+ = \left[\frac{I-I_z+1}{I+I_z+1}\right]^{1/2} b_+$$

$$C_- = -\left[\frac{I+I_z}{I-I_z}\right]^{1/2} b_-$$

we get that the constraint equations become $I = I_z = 0$ and

$$\langle \varphi^{00Y} \rangle \neq 0$$

Also

$$Y \neq 0 \qquad \text{and} \qquad b_-\langle \varphi^{I-\frac{1}{2};I_z+\frac{1}{2};Y+1} \rangle \neq 0$$

From these constraints we get

$$\frac{\mu-\lambda}{3}+\tfrac{1}{2}-\tfrac{1}{2}Y \neq 0$$

$$\frac{\lambda+2\mu}{3}+\tfrac{1}{2}+\tfrac{1}{2}Y \neq 0$$

and

$$\frac{2\lambda+\mu}{3}+\tfrac{3}{2}-\tfrac{1}{2}Y \neq 0$$

provided $Y \neq -1$.

Combining these results we conclude that the allowed values of $(\lambda-\mu)/3$ are ± 1 and ± 2. This would mean that only these types of $SU(3)$ representations are to be used for the Higgs meson in this model.

Finally the $SU(3)$ case treated graphically by Lieberman may also be considered here. In that model, H is a one-parameter subgroup generated by the electric charge operator Q. Our constraint equations are:

$$\langle \varphi^{II_zY} \rangle \neq 0$$

for $Y = -2I_z$; $U_z \neq 0$; $Y \neq 0$; $I_z \neq 0$ and $I = 0$. The allowed representations are then given by the condition: $b_\pm \neq 0$ for $Y + 1 \neq 0$; $I_z + \frac{1}{2} \neq 0$ and $I \pm \frac{1}{2} = 0$.

Substituting into the expressions for b_+ and b_- one gets that the only allowed representations are of the form

$$\frac{\lambda - \mu}{3} = 0, \pm 1, \pm 2$$

Thus representations like $D^1(0, 0), D^8(1, 1), D^{10}(3, 0), D^{27}(2, 2)$ are allowed, while representations like $D^3(1, 0), D^3(0, 1)$ are forbidden. This agrees partially with the graphical results of Lieberman that only the low-dimensional representations are forbidden.

We find that whole class of representations, including the low-dimensional quark representations $D^3(1, 0)$ and $D^3(0, 1)$, are forbidden. It is, however, interesting to note that the physically relevant representations like $D^8(1, 1)$, $D^{10}(3, 0), D^{10}(0, 3), D^{27}(2, 2)$ etc. are all allowed.

3. *Conclusion*

We conclude that in models of spontaneously broken gauge symmetries, the representation of the gauge group to which the Higgs meson may be assigned is not completely arbitrary. Certain representations will lead to mathematical inconsistencies.

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